Effective Support Size

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1. Effective size of support

Let X be a discrete random variable which can take on values from a finite set $\mathcal{X}$ of m elements, with probabilities specified by the probability mass function (pmf) p. The support of X is a set $\mathcal{S}(p(X)) \triangleq \{p : p_i > 0, i = 1, 2, \ldots, m\}$. Let $|\mathcal{S}(p(X))|$ denote the size of the support.

While pmf $p = [0.5, 0.5]$ makes both outcomes equally likely, the following pmf $q = [0.999, 0.001]$ characterizes a random variable that can take on almost exclusively only one of two values. However, both p and q have the same size of support. This motivates a need for a quantity that could measure size of support of the random variable in a different way, so that the random variable can be according to its pmf placed in the range $[1, m]$. We will call the new quantity/measure the effective support size (Ess), and denote it by $\mathbb{S}(p(X))$, or $\mathbb{S}(p)$, for short. The example makes it obvious that $\mathbb{S}(\cdot)$ should be such that $\mathbb{S}(q)$ will be close to 1, while to p it should assign value $\mathbb{S}(p) = 2$.

2. Properties of Ess

Ess should have certain properties, dictated by common sense.

P1) $\mathbb{S}(p)$ should be continuous, symmetric function.

P2) $\mathbb{S}(\delta_m) = 1 \leq \mathbb{S}(p_m) \leq \mathbb{S}(u_m) = m$; where u_m denotes uniform pmf on m-element support, δ_m denotes an m-element pmf with probability concentrated at one point, p_m denotes a pmf with $|\mathcal{S}(p)| = m$.

P3) $\mathbb{S}([p_m, 0]) = \mathbb{S}(p_m)$.

P4) $\mathbb{S}(p(X, Y)) = \mathbb{S}(p(X))\mathbb{S}(p(Y))$, if X and Y are independent random variables.

The first two properties are obvious. The third one states that extending support by an impossible outcome should leave Ess unchanged. Only the fourth property needs, perhaps, some little discussion. Or, better, an example. Let $p(X) = [1, 1, 1]/3$ and $p(Y) = [1, 1]/2$ and let X be independent of Y. Then $p(X, Y) = [1, 1, 1, 1, 1, 1]/6$. According to P2), $\mathbb{S}(p(X)) = 3$, $\mathbb{S}(p(Y)) = 2$ and $\mathbb{S}(p(X, Y)) = 6 = \mathbb{S}(p(X))\mathbb{S}(p(Y))$. It is reasonable to require the product relationship to hold for independent random variables with arbitrary distributions.

The properties P1)-P4) are satisfied by $\mathbb{S}(p, \alpha) = (\sum_{i=1}^{m} p_i^\alpha)^{\frac{1}{1-\alpha}}$, where α is positive real number, different than 1. Note that $\mathbb{S}(\cdot)$ of this form is exp

Table 1: $\mathbb{S}(p, \alpha)$ for $\alpha = 0.001, 0.1, 0.5, 0.9, 1.0, 1.5, 2.0, 10, \infty$ and different p's.

α	$[0.5, 0.5]$	$[0.6, 0.4]$	$\mathbb{S}(p, \alpha)$ $[0.7, 0.3]$	$[0.8, 0.2]$	$[0.9, 0.1]$	$[1.0, 0.0]$
0.001	2.000000	1.999959	1.999826	1.999554	1.998979	1.000000
0.1	2.000000	1.995925	1.982696	1.956233	1.902332	1.000000
0.5	2.000000	1.979796	1.916515	1.800000	1.600000	1.000000
0.9	2.000000	1.964013	1.856116	1.675654	1.416403	1.000000
1.0	2.000000	1.960132	1.842023	1.649385	1.384145	1.000000
1.5	2.000000	1.941178	1.777878	1.543210	1.275510	1.000000
2.0	2.000000	1.923077	1.724138	1.470588	1.219512	1.000000
10.0	2.000000	1.760634	1.486289	1.281379	1.124195	1.000000
∞	2.000000	1.666666	1.428571	1.250000	1.111111	1.000000

of Rényi's entropy. For $\alpha \to 1$, $\mathbb{S}(p, \alpha)$ also satisfies P1)-P4) and takes the form of $\exp(H(p))$, where $H(p) \triangleq -\sum_{i=1}^{m} p_i \log p_i$ is Shannon's entropy. It is thus reasonable to define $\mathbb{S}(p, \alpha)$ for $\alpha = 1$ this way (with the convention $0 \log 0 = 0$), so that $\mathbb{S}(\cdot)$ then becomes a continuous function of α.

3. Selecting α

The requirements define entire class of measures of effective support size. This opens a problem of selecting α. In Table 1, $\mathbb{S}(p, \alpha)$ is given for various two-element pmf's, and $\alpha = 0.001, 0.1, 0.5, 0.9, 1.0, 1.5, 2.0, 10, \infty$. The value $\mathbb{S}(p, \alpha \to \infty)$ can be found analytically.

From the table it can be seen that the smaller the α, the more $\mathbb{S}(\cdot, \alpha)$ ignores the actual difference between probabilities. For $p = [0.9, 0.1]$ the difference is 0.8, yet $\mathbb{S}(p, 0.001) = 1.998979$, i.e., it interprets the pmf as being very close to $[0.5, 0.5]$.

Based on the table, we would opt for $\mathbb{S}(\cdot, \alpha \to \infty)$ as the good measure of Ess. However, for larger $|\mathcal{S}|$ this choice becomes less attractive. This can be seen easily from a consideration of continuous random variables.

4. Selecting α: continuous case

In the case of continuous random variable $\mathbb{S}(f(x), \alpha) \triangleq (\int f^{\alpha}(x) dx)^{\frac{1}{1-\alpha}}$. For gaussian $n(\mu, \sigma^2)$ distribution, $\mathbb{S}(\cdot, \alpha) = \frac{\sqrt{2\pi\sigma^2}}{\alpha^{\frac{1}{2(1-\alpha)}}}$; cf. [3]. This for $\alpha \to \infty$ converges to $\sqrt{2\pi\sigma^2}$, so that for $\sigma^2 = 1$ it becomes $\sqrt{2\pi} = 2.5067$. It is worth comparing with $\mathbb{S}(\cdot, \alpha = 1) = \sqrt{2e\pi\sigma^2}$ (cf. [1]), which reduces in the case of $\sigma^2 = 1$ to 4.1327. This makes much more sense.

That $\mathbb{S}(\cdot, \alpha \to \infty)$ is not the appropriate measure of Ess can be even more clearly seen in the case of the Exponential distribution. For $\beta e^{-\beta x}$ with $\beta = 1$, $\mathbb{S}(\cdot, \alpha \to \infty) = 1$ while $S(\cdot, \alpha = 1) = e$.

5. Adding another property

The above considerations suggest that $\mathbb{S}(\cdot, \alpha = 1)$ might be the most appropriate of the Ess measures which satisfy the requirements P1)-P4). The question is whether there is some other requirement that is reasonable to add to the already employed properties, such that it could narrow down the set of feasible $\mathbb{S}(\cdot, \alpha)$ to $\mathbb{S}(\cdot, 1)$.

To this end, let us consider two random variables X, Y that are dependent. Let $p(Y|X)$ be the conditional distribution and $p(X, Y)$ the joint distribution. For any of them its Ess can be obtained by $\mathbb{S}(\cdot, \alpha)$. For instance, let X can take on two values x_1, x_2. Then, $\mathbb{S}(p(Y|X = x_1), \alpha)$ is Ess of the conditional distribution of Y given that X has taken the value x_1.

In analogy with P4) it seems reasonable to define Ess for a mean of the conditional distributions $\mathbb{S}(\overline{p}(Y|X), \alpha)$ as $\mathbb{S}(\overline{p}(Y|X), \alpha) \triangleq \frac{\mathbb{S}(p(X,Y),\alpha)}{\mathbb{S}(p(X),\alpha)}$. Note that $\mathbb{S}(\overline{p}(Y|X), \alpha)$ is the same regardless of what value the conditioning variable X has taken. This is why it is a kind of Ess for a mean of the conditional distributions. Note also that when X and Y are independent the definition reduces to the requirement P4).

Now, once the new object is defined, one might wonder whether it can be related to Ess's of the conditional distributions. For $\alpha = 1$ such a relationship indeed exists:

$$\mathbb{S}(\overline{p}(Y|X), 1) = \prod_{i=1}^{n} \mathbb{S}(p(Y|X = x_i), 1)^{p(X=x_i)}. \qquad (1)$$

If Eq. (1) was turned into the fifth requirement, then by invoking Khinchin's [2] uniqueness theorem (which characterizes Shannon's entropy), it can be claimed that $\mathbb{S}(\cdot, 1)$ is the only Ess which satisfies the enhanced set of requirements.

It should be added, however, that Eq. (1) is not the only perceivable relationship between $S(\overline{p}(Y|X))$ and Ess's for conditional distributions. Instead of the form of weighted geometric mean the relationship could for instance take the form of weighted arithmetic mean. Whether in this case there is some α which could satisfy the relationship remains to be an open problem (at least for the present author).

6. Summary

In this speculation we entertained the newly-introduced concept of effective support size (Ess). There are some obvious requirements P1)-P4) that

Ess has to satisfy. The class of Ess measures $\mathbb{S}(\cdot.\alpha) = (\sum_{i=1}^{m} p_i^{\alpha})^{\frac{1}{1-\alpha}}$ which satisfies the requirements is broad. The Ess measures are in a direct relationship to the family of Rényi's entropies which includes as its special case also Shannon's entropy. We have briefly addressed the issue of selecting α such that the corresponding $\mathbb{S}(\cdot, \alpha)$ would be the most 'appropriate' measure of Ess. The considerations indicate that $\alpha = 1$ could, perhaps, be the most reasonable candidate. If Eq. (1) was added into the set of requirements, then $\mathbb{S}(\cdot, 1)$ would become the only $\mathbb{S}(\cdot)$ that satisfies them. However, there are also other conceivable relationships between $S(\overline{p}(Y|X))$ and the conditional $\mathbb{S}(\cdot)$. Whether some of them could be satisfied by $\mathbb{S}(\cdot, \alpha)$ for some other α remains to be an open question. In any case, with the concept of Ess it is possible to enter a meaningful world which is in a sense dual to that of entropies.

References

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